Strategic Voting in Multi-Winner Elections with Approval Balloting: A Theory for Large Electorates*

Jean-François Laslier
Paris School of Economics & CNRS

Karine Van der Straeten
Toulouse School of Economics, CNRS & Institute for Advanced Study in Toulouse

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Abstract

We propose a theory of strategic voting in multi-winner elections with approval balloting: A fixed number M of candidates are to be elected; each voter votes for as many candidates as she wants; the M candidates with the most votes are elected. We assume that voter preferences are separable and that there exists a tiny probability that any vote might be misrecorded.

Best responses involve voting by pairwise comparisons. Two candidates play a critical role: the *weakest expected winner* and the *strongest expected loser*. Expected winners are approved if and only if they are preferred to the strongest expected loser and expected losers are approved if and only if they are preferred to the weakest expected winner.

At equilibrium, if any, a candidate is elected if and only if he is approved by at least half of the voters. With single-peaked preferences, an equilibrium always exists, in which the first M candidates according to the majority tournament relation are elected.

The theory is tested on individual data from the 2011 Regional Government election in Zurich.

1 Introduction

In many instances, societies choose, by voting, a group of representatives. Voting rules for these kinds of elections are more complex than rules designed to

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elect one and only one candidate, and are much less studied in the theoretical literature. Cases of interest include Parliamentary elections and committee selection.

Parliamentary elections in majoritarian systems often proceed by dividing the electorate in subgroups, usually on a geographical basis, and by electing one or several MPs in each such district (Blais and Massicotte 2002, Pukelsheim 2014). The number of delegates is usually fixed for each district, although it may be adjusted in view of the overall results (such is the case in Germany). The set of candidates can be structured with party lists or can be composed of independent/individual candidates.

Committee selection, where a fixed-sized committee has to be elected, offers another example. Note that, in the case of a committee, another kind of complexity may arise because, contrary to most Parliamentary elections, some structure is often imposed on the set of elected candidates. For example the chosen committee must reflect some gender or status balance.

The present paper will concentrate on the simplest case where: *(i)* the number of candidates to be elected is fixed, *(ii)* there is no constraint on the structure of the set of elected candidates, *(iii)* the electorate is not divided and the voters are anonymous (Elkind et al. 2014). Then a natural rule is that each voter can vote for several candidates and the candidates with the largest numbers of votes are elected. Under "Unrestricted Approval Voting", each voter can vote for as many candidates as she wishes, giving at most one vote to each candidate (no "cumulative voting"). Under "Restricted Approval Voting", a voter cannot cast more than a fixed number of votes (usually set to the number of candidates to be elected, but this needs not be). The contribution of this paper is twofold. First, since there exists so far no complete and testable theory of strategic voting at the individual level (best-responses) in multi-winner elections with approval balloting, we propose such a theory, drawing on previous works for the standard case of Approval voting for electing a single candidate (Laslier 2009, Nuñez 2010a). Second, we derive equilibrium predictions.

So far, the literature on multi-winner elections with approval balloting has mainly focused on the different ways approval-type ballots can be counted for electing a committee (of fixed size or not). Electing the candidates with the largest approval scores is the simplest but not the only idea one can have (Fishburn 1981; Aziz et al. 2015). Kilgour (2010) surveys the many proposals which have been made, and Laffond and Lainé (2010) survey the representativeness issue under an assumption of separable preferences. This issue is often tackled in the theoretical literature under the assumption that the committee size is not fixed, which makes the problem similar to a multiple referendum problem. In this vein, see Gehrlein (1985), Bock et al. (1998), Brams et al. (1997, 1998), Brams et al. (2007). We here focus on the case —often met in practice— of a fixed-size committee.

The issue of the voter's behavior (which ballot to cast?) is not addressed by the previously mentioned studies and one question left pending is to describe "sincere" and "rational" behavior in these elections and to evaluate the level of strategic voting induced by such a voting rule. One exception is Cox (1984) who studies the special case of multi-member districts with two members to be elected and three candidates, when the voter is allowed to cast up to two votes. He shows that depending on the context (anticipations about other voters' behavior and own preferences), strategic voting in such an election entails either voting for one's preferred candidate only, or voting for one's two preferred candidates. In this paper, we will characterize best-responses for any configuration about the number of candidates to be elected, the number of candidates, and the maximal number of votes a voter is allowed to cast. We will also consider equilibrium predictions.

Section 2 describes the model. We will assume that voter preferences over committees are separable and that there exists a tiny probability that any vote might be mis-recorded. This latter assumption will guarantee that the voter is uncertain about the realized scores of the candidates, even when she knows other voters' strategies (as is standard in strategic voting models, see for example Myerson and Weber 1993). Section 3 provides some preliminary results on the probability of some critical pivot events. Focusing first on the case of "Unrestricted Approval Voting", Section 4 studies best responses and Section 5 studies equilibria. Section 6 is devoted to the case where a limit is set on the number of candidates a voter can approve ("Restricted Approval Voting"). Section 7 tests the theory on real data gathered during an election of the regional government of the canton of Zurich (composed of seven members), in Switzerland, where the voting rule is essentially the one we study in theory. We show that roughly 70% of the individual decisions on candidates are consistent with our model of rational voting, and identify the main remaining point of discrepancy between this theory and the observations. Section 8 concludes. Long proofs are relegated in an appendix.

2 A model of multi-winner elections with approval balloting

In the sequel, although our analysis equally applies to some Parliamentary elections as well as to committee selection, we will use the committee terminology, and will, for example, talk about the size of the committee when referring to the number of candidates to be elected.

We first study (Sections 2 to 5) the case of "Unrestricted Approval Voting", where each voter can vote for as many candidates as she wishes, with no limit on the number of votes she can cast. The case of "Restricted Approval Voting" will be tackled in Section 6.

(Unrestricted) Approval Voting M seats have to be filled. The set of candidates, of size $K > M$, is denoted by $\mathfrak{C}$. There are N voters, $i = 1, ..., N$.

Voters vote by casting votes for candidates; they can give at most one vote to a candidate (no "cumulative voting") but can vote for several candidates. The M candidates with the highest numbers of votes are elected. Ties, if any, are randomly broken.

Voter preferences Voters preferences over committees are supposed to be separable across candidates in the following sense: Voter i has a utility function u_i for candidates, and the utility for the committee C is the sum $\sum_{c\in C} u_i(c)$, where C is any subset of size M of the set of candidates $\mathfrak{C}$. We assume for simplicity that preferences over the set of candidates (as described by the utility function u_i) are strict.

Preferences are common knowledge and there is no uncertainty about the size (N) of the electorate.

Voter strategies For $i = 1, ..., N$, a strategy for voter i is a vector

$$s_i = (s_{i,c})_{c\in\mathfrak{C}} \in \{0,1\}^K ,$$

where for all c, $s_{i,c} = 1$ if voter i casts a vote in favor of candidate c , and $s_{i,c} = 0$ if voter i does not cast a vote for candidate c (we will also use the terminology "casts a vote against candidate c" or simply "votes against c").

Small mistakes As is standard in strategic voting models (see for example Myerson and Weber 1993), we assume that the voter is uncertain about the realized scores of the candidates, even when she knows other voters' strategies. Uncertainty is modelled as follows. As described above, preferences are common knowledge and there is no uncertainty about the size (N) of the electorate. But, for any vote which is actually cast for a candidate by a voter, there is a tiny possibility of mistake, a mistake resulting in that vote not being recorded. Conversely, even if a voter has not voted for a candidate, there is a tiny probability that this is wrongly recorded as a vote. We assume that the mistakes are made independently across voters and across candidates.[1]

More formally: We suppose that there exists a number $\varepsilon > 0$ such that, for each ballot cast by a voter, and for each candidate c:

- if i votes for c, this vote is recorded with probability $(1-\varepsilon)$, and with probability ε this vote is not recorded;
- if i does not votes for c, this is correctly recorded with probability $(1-\varepsilon)$, and with probability ε a vote for candidate c is instead recorded.

For example, with $K = 3$ candidates, assume that a voter has cast the ballot $(1, 1, 0)$. Given our assumptions about the small mistakes made while recording the votes, this ballot is correctly recorded as such with a probability $(1-\varepsilon)^3$, it is recorded as $(0, 1, 0)$ with probability $(1-\varepsilon)^2 \cdot \varepsilon$ (one mistake), ..., and recorded as $(0, 0, 1)$ with probability ε^3 (three mistakes).

These assumptions guarantee that, for any profile of ballots cast by the voters, all electoral outcomes (realized scores of candidates) have a positive probability.

[1] There is no independence across candidates in Nuñez (2010b).

Voters' beliefs Preferences and the structure of the game, including the possibilities of mistakes described above, are common knowledge among the voters.

We will assume that the voters in their computation of best responses neglect the possibility of three-way ties; a cognitive assumption which seems realistic for an individual taking part to a large election. When needed, we also assume that the expected scores of any two candidates differ by at least three votes.

These assumptions are well suited for large elections (typically, political ones) but would not be reasonable if one wanted to study small electorates.

3 Pivotal events with minimal requirement

In order to determine her best response against the other voters' strategies, the voter will have to estimate the probability of the different events where her vote might be pivotal (that is, change the outcome of the election). Before turning to the study of best responses (Section 4) and equilibria (Section 5), we start by computing the order of magnitude of some critical pivot events. We will first introduce the notion of *minimal requirement*, then use it to estimate the order of magnitude of some critical events involving ties between candidates (Lemma 1 and Lemma 2).

Distribution of realized scores For a profile of ballots $s = (s_i)_{i=1,\ldots,N}$ and for a candidate c, denote by $\widehat{s}(c) = \sum_i s_{i,c}$ the number of voters who vote for c, and by $S(c)$ the random variable describing the realized score of candidate c (taking into account the possibility of mistakes) obtained from these ballots.

For any two candidates c and c', $S(c)$ and $S(c')$ are independent random variables, with expectations $\widehat{s}(c)$ and $\widehat{s}(c')$ respectively. Note that the random variable $S(c)$ can be written as:

$$S(c) = \sum_i \left[s_{i,c}(1-\omega_{i,c}) + (1-s_{i,c})\,\omega_{i,c}\right], \tag{1}$$

where the $\omega_{i,c}$, for $i = 1, ..., N$ and $c \in \mathfrak{C}$, are $N \times K$ independent random draws which take value 0 with probability $(1-\varepsilon)$ and 1 with probability ε. Here, $\omega_{i,c} = 1$ means that a mistake is made when recording voter i's vote about candidate c. We will call ω an "elementary event." The probability of ω is:

$$\Pr[\omega] = \varepsilon^{|\omega|} \times (1-\varepsilon)^{NK-|\omega|},$$

where $|\omega| = \sum_{i,c} \omega_{i,c}$ denotes the total number of mistakes associated to the elementary event ω. Since $(1-\varepsilon)^{NK-|\omega|} = \sum_{k=0}^{k=NK-|\omega} \binom{NK-|\omega|}{k}(-\varepsilon)^k$, one gets that:

$$\Pr[\omega] = \sum_{k=0}^{NK-|\omega|} (-1)^k \binom{NK-|\omega|}{k} \varepsilon^{|\omega|+k}. \tag{2}$$

Notice that this is a polynomial in ε, whose first term of lowest degree is $\varepsilon^{|\omega|}$: when the probability of mistake ε goes to zero, the probability of the elementary event ω is asymptotically equivalent to $\varepsilon^{|\omega|}$.

Definition of the "requirement of an event" Fixing a profile of ballots s, all (aggregate) electoral outcomes, that is, all possible vectors of scores (for the candidates), occur with positive probability (see expression (1) for the realized score of candidate c). We define an event as a subset of this set of all possible electoral outcomes. Any event E can be expressed with the help of the elementary events ω and thus has a probability which is a polynomial in ε. For any event E, let us denote by $A(E)\varepsilon^{m(E)}$ the term of lowest degree of this polynomial, where $m(E)$ is the smallest number of mistakes required to realize E, and $A(E)$ is the number of ways to realize E with $m(E)$ mistakes (given s). The exponent $m(E)$ will be called the **requirement** of event E. Note that, from (2), for any elementary event $\omega \in \{0,1\}^{N\times K}$, $m(\omega) = |\omega|$ and $A(\omega) = 1$.

The requirement of an event is an indicator of how unlikely this event is to happen. Indeed, between two events E and E' with requirements m and m' respectively, with $m < m'$, the probability of E' is "vanishingly small" compared to the probability of E, meaning that when ε tends to 0, the ratio $\Pr[E']/\Pr[E]$ tends to 0. This concept of requirement will play an important role when deriving best responses.

Computation of the requirement of critical events The following two lemmas will give some insights about the requirement of some critical events involving ties between candidates.

Lemma 1 *Given a profile of strategies (ballots) $s = (s_i)_{i=1,\dots,N}$, for any two candidates c and c', the requirement of the event "$S(c) = S(c')$" is $|\widehat{s}(c) - \widehat{s}(c')|$.*

Lemma 1 states that, given the ballots cast by the voters, the probability of candidates c and c' obtaining the exact same realized scores is asymptotically equivalent to $\varepsilon^{|\widehat{s}(c)-\widehat{s}(c')|}$, where the exponent $|\widehat{s}(c) - \widehat{s}(c')|$ is the absolute value of the difference in expected scores between candidates c and c'. The proof of the lemma is provided in the appendix (section A.1).

Consider now any candidate c. We will say that realized scores are such that candidate c is *caught in an exact tie for election* if whether c is elected or not has to be determined by a random draw (at least two candidates, including c, tie for the M-th position). The following lemma provides the requirement of such an event, for all candidates.

Lemma 2 *Given a profile of strategies $s = (s_i)_{i=1,\dots,N}$, assume that candidates are labelled in such a way that:*

$$\widehat{s}(c_1) > \widehat{s}(c_2) > \dots > \widehat{s}(c_M) > \widehat{s}(c_{M+1}) > \dots > \widehat{s}(c_K).$$

(i) If $k \leq M$, the requirement of the event "Candidate c_k is caught in an exact tie for election" is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$. Besides, any event of minimal requirement where c_k is caught in an exact tie for election involves a tie with candidate c_{M+1}.

(ii) If $k \geq M+1$, the requirement of the event "Candidate c_k is caught in an exact tie for election" is $\widehat{s}(c_M) - \widehat{s}(c_k)$. Besides, any event of minimal requirement where c_k is caught in an exact tie for election involves a tie with candidate c_M.

Note the crucial role played by two candidates: c_M and c_{M+1}. The former is the candidate whose expected score is the M-th largest — we will call this candidate the *weakest expected winner* and the latter the candidate whose expected score is the $(M+1)$-th largest — we will call this candidate the *strongest expected loser*. The proof of the lemma is provided in the appendix (section A.2).

4 Best responses

4.1 Characterization

We first describe a voter's, say voter i's, best response against a profile of strategies $s_{-i} = (s_j)_{j \neq i}$ by the other $N-1$ voters. Given this profile s_{-i}, for all c, denote by $\widehat{s}_{-i}(c) = \sum_{j \neq i} s_{j,c}$ the number of voters (other than voter i) who vote for c. Given our model of uncertainty, $\widehat{s}_{-i}(c)$ is the expected score of candidate c, not taking into account the vote of voter i. Proposition 3 describes the voter's best response in the case where the expected vote difference between any two candidates is at least 3.

Proposition 3 *Let $\widehat{s}_{-i}$ denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i. Let the candidates be labelled in such a way that:*

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > ... > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > ... > \widehat{s}_{-i}(c_K). \tag{3}$$

Assume that the expected vote difference between any two candidates is at least 3, that is, for any pair of candidates (c, c'), $|\widehat{s}_{-i}(c) - \widehat{s}_{-i}(c')| \geq 3$.
For ε small enough, the best response of voter i is the following:

- *For $1 \leq k \leq M$: Voter i votes for c_k if and only if $u_i(c_k) > u_i(c_{M+1})$,*
- *For $M+1 \leq k \leq K$: Voter i votes for c_k if and only if $u_i(c_k) > u_i(c_M)$.*

With assumption (3) regarding the ranking of candidates, the first M candidates are the expected winners, and the other candidates are the expected losers.[2]

[2] The additional assumption that the expected vote difference between any two candidates in $\widehat{s}_{-i}$ is at least 3 guarantees that the expected winners and losers in the election remain the same whatever the ballot chosen by voter i.

Proposition 3 states that the voter should vote for an expected winner if and only if she prefers that candidate to the candidate ranked $M + 1$, that is, the *strongest expected loser*. Symmetrically, the voter should vote for an expected loser if and only if she prefers that candidate to the candidate ranked M, that is, the *weakest expected winner*. Best responses are thus quite easy to describe: they entail voting by pairwise comparison with those two critical candidates: the strongest expected loser and the weakest expected winner.

The proof of the proposition is presented in the appendix (A.3) but the intuition is quite simple. It mostly derives from Lemma 2, which states that the requirement of the event "Candidate c_k is caught in an exact tie for election" is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ if $k \leq M$ and $\widehat{s}(c_M) - \widehat{s}(c_k)$ if $k \geq M + 1$.[3]

- Therefore, the most likely tie for election occurs between candidates c_M (the weakest expected winner) and c_{M+1} (the strongest expected loser), since the requirement of this event is $\widehat{s}(c_M) - \widehat{s}(c_{M+1})$. If voter i is pivotal, it will most likely be in deciding who between candidate c_M and candidate c_{M+1} will be elected.[4] Therefore, if she prefers candidate c_M to candidate c_{M+1} ($u_i(c_M) > u_i(c_{M+1})$), she should vote for candidate c_M and not vote for candidate c_{M+1}. Similarly, if $u_i(c_M) < u_i(c_{M+1})$, she should vote for candidate c_{M+1} and not vote for candidate c_M. Her decision about candidates c_M and c_{M+1} is thus decided by this pairwise comparison between the two candidates.

- Consider now candidate c_k, $k < m$, an expected winner. If candidate k is caught in a tie, it will most likely be against candidate c_{M+1} (Lemma 2). Therefore the voter should vote for candidate c_k if and only if $u_i(c_k) > u_i(c_{M+1})$: the vote for candidate c_k if decided by a pairwise comparison with the strongest expected loser c_{M+1}.

- Similarly, if candidate c_k, $k > M + 1$, is caught in a tie, it will most likely be against candidate c_M, and therefore the voter should vote for candidate c_k if and only if $u_i(c_k) > u_i(c_M)$: the vote for candidate c_k if decided by a pairwise comparison with the weakest expected winner c_M.

4.2 Properties of best responses: Sincere and non-sincere voting

According to the usual definition in the Approval Voting literature (Brams 1982), a ballot is "sincere" for a voter if, when the voter approves a candi-

[3] Lemma 2 takes into account the votes of all voters, including voter i. To derive the best response of voter i, the argument has to be adjusted to take into account the fact that voter i takes the votes of other voters as given, but not her. These adjustments are made in the proof in the appendix, but the intuition about the orders of magnitude of the different pivot events remains similar.

[4] Lemma 2 deals with exact ties for election. A voter can also be pivotal in case of a near tie (one vote margin) for election between two candidates. Noting that a requirement of a near tie is no larger than the requirement of an exact tie plus one, the arguments carry through when explicitly taking into account the possibility of near ties (which is done in the proof).

date c, she also approves all the candidates she strictly prefers to c. Proposition 4 characterizes the parameters of the electoral context such that a best response always entails casting a sincere ballot, whatever the voter's preferences and the other voters' strategies.

Proposition 4 *Consider the best response function described in Proposition 3, for* $\varepsilon \to 0$.

- *If* $M = 1$ *or* $K = M+1$, *the best response always entails casting a* ***sincere ballot****, whatever the voter's preferences and the other voters' strategies.*
- *Otherwise, there exist voter's preferences and other voters' strategies, such that the best response entails casting a* ***non-sincere ballot****.*

Laslier (2009) noticed that a best response always entails sincere voting when there is one single candidate to elect ($M = 1$). Cox (1984) noticed that a best response always entails sincere voting when there are two candidates to be elected from a set of three candidates ($M = 2$ and $K = 3$).[5] The first point of the Proposition generalizes this result to the case where the number of running candidates exceeds only by one the number of candidates to be elected ($M = K - 1$).

But it also shows that these two cases ($M = 1$ and $M = K - 1$) are rather specific in the sense that in any other configuration about the number of seats and the number of candidates, there will be situations (preferences and anticipations about other voters' behavior) such that strategic voting is non-sincere.

The intuition for the existence of non-sincere ballots is that the strategic recommendation entails voting by pairwise comparisons, but that expected winners and expected losers are compared to two different candidates (the strongest expected loser and the weakest winner, respectively). Note that if all candidates where compared to the same benchmark, sincere voting would result (as is basically the case when $M = 1$: all candidates —but himself— are compared to the expected winner).

The proof of the proposition is in the appendix (section A.4).

5 Equilibrium

5.1 Characterization

Let us now study the nature of equilibria consistent with the strategic behavior described in Proposition 3. For simplicity, we assume that N is odd .

[5] To be precise, the voting rule studied by Cox was slightly different from the one considered here, since voters are only allowed to cast up to two votes ("Restricted Approval Voting"). Yet, it is straightforward to check that strategic voting implies never voting for one's least prefer candidate, therefore, when there are only three candidates, a best response entails casting at most two votes. The two rules are therefore equivalent from a strategic point of view, for three candidates.

Denote by $N(c,c')$ the number of voters who prefer[6] candidate c to candidate c'. For $i=1,...,N$, denote by $N_{-i}(c,c')$ the number of voters, other than voter i, who prefer candidate c to candidate c'. We assume that for all i and for all (c,c'), (c'',c'''), with $(c,c')\neq(c'',c''')$, the following condition is satisfied: $|N_{-i}(c,c')-N_{-i}(c'',c''')|\geq 3$. Clearly, this is not totally general. But this simplification is reasonable when the number of voters is large. The following characterization of an equilibrium will be useful in the sequel.

Proposition 5 *A profile of strategies $(s_i)_{i=1,...,N}$ is a* pure equilibrium *if and only if there exists a partition of the set of candidates into two candidates (call them c_M and c_{M+1}) and two subsets of candidates, $\{c_1,...,c_{M-1}\}$ and $\{c_{M+2},...,c_K\}$ such that:*

1. $N(c_M,c_{M+1})>N(c_{M+1},c_M)$,
2. $k<M\Longrightarrow N(c_k,c_{M+1})>N(c_M,c_{M+1})$,
3. $k>M+1\Longrightarrow N(c_k,c_M)<N(c_{M+1},c_M)$,
4. *For $i=1,...,N$, s_i is the best response described in Proposition 3 against the expected scores (from the $N-1$ other voters) defined as follows:*

$$\begin{aligned}\widehat{s}_{-i}(c_k) &= N_{-i}(c_k,c_{M+1}) \text{ if } k\leq M,\\ \widehat{s}_{-i}(c_k) &= N_{-i}(c_k,c_M) \text{ if } k\geq M+1.\end{aligned}$$

Then the (expected) winners are the members of the set $\{c_1,...,c_M\}$. The expected scores are $\widehat{s}(c_k)=N(c_k,c_{M+1})$ if $k\leq M$ and $\widehat{s}(c_k)=N(c_k,c_M)$ if $k\geq M+1$.

This characterization makes clear a strong link between approval voting for a committee and a notion of "majority rule", as noted in the following remark, whose transparent proof is provided.

Remark 6 *In a pure equilibrium (if any):*
(i) A candidate is an expected winner if and only if he is approved by at least half of the voters,
(ii) The Condorcet winner (if it exists) is an expected winner.

Proof. Part (i). Consider an equilibrium, where the (expected) scores of the candidates are:

$$\widehat{s}(c_1)>\widehat{s}(c_2)>...>\widehat{s}(c_M)>\widehat{s}(c_{M+1})>...>\widehat{s}(c_K).$$

From the characterization in Proposition 5, the expected scores are $\widehat{s}(c_k)=N(c_k,c_{M+1})>N(c_M,c_{M+1})$ if $k<M$ and $\widehat{s}(c_k)=N(c_k,c_M)<N(c_{M+1},c_M)$ if $k>M+1$.

[6] Remember we assume strict preferences over the set of candidates.

From condition (1) in the characterization of a pure equilibrium, we know that: $N(c_M, c_{M+1}) > N(c_{M+1}, c_M)$. Since $N(c_M, c_{M+1}) + N(c_{M+1}, c_M) = N$, this implies that $\widehat{s}(c_M) = N(c_M, c_{M+1}) > N/2$ and $\widehat{s}(c_{M+1}) = N(c_{M+1}, c_M) < N/2$.

Thus $\widehat{s}(c_k) > N/2$ for all $k \leq M$ and $\widehat{s}(c_k) < N/2$ for all $k \geq M+1$.

Part (ii). At equilibrium, an expected loser has to be defeated in a pairwise comparison with the weakest expected winner. Since the Condorcet winner (when it exists), would defat any other candidate in a pairwise vote, this shows that the Condorcet winner (if it exists) has to be an expected winner. ■

An alternative interpretation: *Trembling hand perfection* There is a strong link between a pure equilibrium with the model of small mistakes introduced in section 2 and the concept of *trembling hand perfect equilibrium.* Trembling hand perfect equilibrium is a refinement of Nash equilibrium due to Selten (1975). A trembling hand perfect equilibrium is an equilibrium that takes the possibility of off-the-equilibrium play into account by assuming that the players, through a "slip of the hand" or tremble, may choose unintended strategies, albeit with small probability. One may check that a pure equilibrium in our game with recording mistakes is a trembling hand perfect equilibrium in the game with no recording mistakes (with trembles occurring with probabilities consistent with the model of small mistakes described here).

5.2 Existence and uniqueness

The following two remarks provide the theoretical answers to the questions of existence and uniqueness of equilibrium.

Remark 7 ***Non existence of equilibrium.*** *Whenever $K > M+1$, there may exist no pure equilibrium.*

Proof. Take $M = 1$. It is easy to check that a pure equilibrium exists if and only if there exists a Condorcet winner. Indeed, from the characterization of equilibrium above (Proposition 5), there must exist some candidates c_1, c_2 such that conditions 1 and 3 are satisfied (condition (2) is empty). Condition (1) yields: $N(c_1, c_2) > \frac{N}{2}$. Condition (3) yields: $k \geq 3 \Longrightarrow N(c_k, c_1) < N(c_2, c_1)$ Since $N(c_2, c_1) < \frac{N}{2}$, one sees that c_1 is a Condorcet winner. Since a Condorcet winner may not exist, there will be profiles of preferences for which there is no pure equilibrium as soon as there are at least three candidates.

For $M \geq 2$, counter-examples are easily found by considering a preference profile with $M-1$ candidates who Pareto-dominates all the others, and no Condorcet winner among the remaining candidates, which is possible as soon as there are at least $M+2$ candidates. ■

Remark 8 ***Multiplicity of equilibria.*** *For $M = 1$, if there is an equilibrium, it is unique. For $M \geq 2$, there may exist several pure equilibria.*

Proof. Take $M = 2$ and $K = 4$. Let a, b, c and d denote the candidates. Consider the following matrix g.

g	a	b	c	d
a	0	4	5	1
b	-4	0	2	6
c	-5	-2	0	3
d	-1	-6	-3	0

We know from Debord (1987) that there exists a preference profile for which the majorities $N(x, y)$ are positive affine transformations of $g(x, y)$. Since our characterization of equilibrium only involves comparisons between the numbers $N(x, y)$, we do not need to know exactly the preference profile and we can simply use the matrix g.

One can check that the following three situations are equilibria:

$$\left(\begin{array}{c} a\ (5) \\ b\ (2) \\ \hline c\ (-2) \\ d\ (-6) \end{array}\right), \left(\begin{array}{c} b\ (6) \\ a\ (1) \\ \hline d\ (-1) \\ c\ (-5) \end{array}\right), \left(\begin{array}{c} c\ (3) \\ a\ (1) \\ \hline d\ (-1) \\ b\ (-4) \end{array}\right).$$

In the first case, a and b are expected winners with respectively 5 and 2 (relative) votes, and c and d are rejected with respectively -2 and -6 votes. These numbers are precisely the pairwise scores of a and b compared to c, and of c and d compared to b. This situation is thus an equilibrium. The reader can check that the other situations, in which the elected candidates are again a and b, or are c and a are also equilibria.

The same example can easily be extended to larger values of M by adding Pareto-dominant candidates.

For $M = 1$, it was proven in the proof of Remark 7 that a pure equilibrium exists if and only if there exists a Condorcet winner. Without indifferences or ties in the vote matrix, there cannot be two Condorcet winners. Denote by c^* the unique Condorcet winner. At equilibrium, the expected score of $c \neq c^*$ is $N(c, c^*)$. Denoting by c_2 the candidate such that $c_2 = \arg\max_{c \neq c^*} N(c, c^*)$, the expected score of c^* is $N(c^*, c_2)$. So that uniqueness of pure equilibrium holds. ■

The two previous remarks raise several questions: How serious are the two problems of non-existence and multiplicity? We have no full answers to these questions in general, but the next section will provide precise answers for a non-trivial domain of preferences.

5.3 Majority-transitive and single-peaked preference profiles

If the majority tournament is transitive, a pure equilibrium exists for any committee size. More exactly the following result holds.

Proposition 9 *Suppose that there exists a set of M candidates such that any candidate in this set beats, according to pairwise-majority voting, any candidate not in this set. Then there exist an equilibrium in which these M candidates are elected.*

Proof. Let C be the set of candidates that beat the others, and let $D = \mathfrak{C} \backslash C$. Let $c \in C$ and $d \in D$ be two candidates such that

$$N(c, d) = \min_{x \in C, y \in D} N(x, y).$$

We will check that the expected scores vector defined by $\widehat{s}(x) = N(x, d)$ for all $x \in C$ and $\widehat{s}(y) = N(y, c)$ for all $y \in D$ is an equilibrium. By definition of C, for all $x \in C$, $\widehat{s}(x) \geq \widehat{s}(c) = N(c, d)$. Likewise, for all $y \in D$, $\widehat{s}(y) = N(y, c) = N - N(c, y) \leq N - N(c, d) = N(d, c)$. Moreover, $N(c, d) > N/2 > N(d, c)$ hence $\widehat{s}$ correctly ranks all the candidates. ■

So existence of equilibrium is guaranteed in that case, but there can be many equilibria. When the majority tournament associated with the preference profile is transitive, the proposition applies to the M first candidates according to the majority tournament order, and thus an equilibrium exists for any M. The example we used previously (Remark 8) to demonstrate the possible multiple equilibria is in fact a transitive tournament, as can be easily seen on the matrix g. Notice that the example shows that different equilibria may not only results in different (expected) scores vectors but also in different elected committees.

A nice application is the case of single-peaked preferences. This point is stated in a separate proposition. Part of it can be derived from the previous one and Remark 6; in the appendix (A.5) we complete a proof which provides a more detailed description of what can and cannot happen for the single-peaked domain.

Proposition 10 *Assume that the candidates can be ordered (in a one-dimensional space) in such a way that voters have single-peaked preferences over the set of ordered candidates. In that case:*

1. *There exists an equilibrium where the first M candidates according to the majority tournament are elected.*
2. *In any equilibrium the Condorcet winner candidate is elected, and the elected committee forms a segment of the ordered set of candidates.*
3. *At most M different committees can be elected at equilibrium.*

Proposition 10 again highlights the strong relationship between equilibrium under strategic approval voting and the majority rule. Remark 6 stated that in a pure equilibrium, all expected winners are approved by a majority of voters. Proposition 10 states than when preferences are single-peaked, the first M

candidates according to the majority tournament are expected winners in some equilibrium.

Proposition 10 states that if voters have single-peaked preferences, there are at most M distinct sets of elected candidates which can be supported at equilibrium. We provide in the appendix (A.5, after the proof of Proposition 10) a simple example showing that this maximum number can be reached: there can be M distinct sets of winners at equilibrium. Besides, in one of these equilibria, among the M elected candidates, only the Condorcet winner belongs to the set of the M top candidates according to the majority tournament.

6 Restricted Approval Voting

This section briefly tackles the rule called "V-Restricted Approval Voting", where a voter can only approve up to V candidates. The case $V = 1$ is thus simple Plurality rule. Unrestricted Approval Voting corresponds to any V larger than K, the number of candidates. The case $V = M$, where the number of votes equals the number of seats, seems natural (it is the one used in the canton of Zurich, on which the theory will be tested in Section 7) but does not seem to have any specific theoretical property, as will be seen.

We keep the same model as in Section 2, with a slight change in the definition of the strategies. For $i = 1, ..., N$, a strategy for voter i is a vector $s_i = (s_{i,c})_{c\in\mathfrak{C}} \in \{0,1\}^K$, such that $\sum_{c\in\mathfrak{C}} s_{i,c} \leq V$, where for all c, $s_{i,c} = 1$ if voter i casts a vote in favor of candidate c , and $s_{i,c} = 0$ if voter i does not cast a vote for candidate c. We keep the description of mistakes made when recording the votes for candidates exactly the same as in Section 2 (in particular, we keep the assumption that mistakes are independent across candidates, meaning that we do not rule out the possibility that strictly more than V (positive) votes are recorded).

6.1 Best responses

Proposition 11 *Let $\widehat{s}_{-i}$ denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i. Let the candidates be labelled in such a way that:*

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > ... > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > ... > \widehat{s}_{-i}(c_K).$$

Assume that the expected vote difference between any two candidates is at least 3, that is, for any pair of candidates (c, c'), $|\widehat{s}_{-i}(c) - \widehat{s}_{-i}(c')| \geq 3$.
For ε small enough, the best response of voter i, when he has at most V votes, can be characterized as follows:

1. *The voter identifies the set of expected winners (c_1 to c_M) and that of expected losers (c_{M+1} to c_K).*

2. *If $1 \leq k \leq M$, define candidate c_k's "main contender" as c_{M+1} (the strongest expected loser) and if $M+1 \leq k \leq K$, define c_k's "main contender" as c_M (the weakest expected loser).*

3. *The voter ranks the candidates according to (the inverse of) their distance, in terms of expected votes, to their main contender.*

4. *The voter considers all the candidates in turn, according to the priority order defined at the previous step. As long as she does not hit the vote-budget constraint (V votes), she votes for a candidate if and only if her utility for this candidate is larger than her utility for its main contender.*

This Proposition is a generalization of Proposition 3, the only difference being, the appearance, in Step 4, of the vote constraint. The proof of Proposition 11 follows the same reasoning as the proof of Proposition 3 (see Section A.3, Remark 12).

As noticed above, the only difference between "Unrestricted Approval" and "Restricted Approval" is the appearance, in Step 4, of the vote constraint. Given the limited number of votes, the voter has to consider the candidates lexicographically, in the order defined in Step 3. In this order, candidates are ranked according to their distance to their most likely contender (in numbers of expected votes). This is equivalent to ranking them by decreasing probability of them being caught in a tie for election. Indeed, as noticed when describing the intuitive content of Proposition 3, the most likely pivot-event is a tie between the two candidates who are expected to rank M-th and $M+1$-th (here candidates c_M to c_{M+1}). What is the next most likely pivot-event? Note that all the other pivot events imply some order reversals among candidates, compared to the expected order. Which is the next pair of candidates between which the voter is most likely to be pivotal? Our assumptions imply that it will be either the pair $\{c_M, c_{M+2}\}$ or the pair $\{c_{M-1}, c_{M+1}\}$, depending on whether the difference in expected scores between c_M and c_{M+2} is larger or smaller than the difference in expected scores between c_{M-1} and c_{M+1}. Indeed, they are the two pairs which require the less order reversals compared to the expected outcome. Similarly, other pivot-events can be ranked by decreasing probability of occurrence.

Note that in that case, there is no reason to expect that the strategic recommendation will entail sincere voting. Indeed, there are now two potential causes as to why the strategic recommendation might not be sincere:

(1) As under "Unrestricted Approval", the expected winners are compared to the strongest expected loser, whereas the expected losers are compared to the weakest expected winner. This fact was exploited to construct counter-examples in the proof of Proposition 4.

(2) The constraint on the number of votes may be binding. The voter, if given the opportunity to cast more votes, would vote for candidates higher in her preferences; but she has used all her votes on candidates with higher probability to be caught in a tie for election. One extreme case is $M = V = 1$ (simple plurality to elect one candidate), where the voter should vote for her

preferred candidate among the two candidates who are expected to receive the most votes: she should desert her preferred candidate whenever he is not one of the two main candidates.

6.2 Equilibrium

Consider the following example with $M = 2$ candidates to be elected, $K = 4$ candidates, and $N = 85$ voters. Denote the candidates by a, b, c, d. The next Table indicates that, for instance, 45 voters prefer a to b, b to c, and c to d.

(45)	(10)	(20)	(10)
$a*$	$b*$	$c*$	$c*$
$b*$	$c*$	$d*$	b
c	a	b	a
d	d	a	d

This preference profile is single-peaked with respect to the order $a < b < c < d$. Assume that the voters vote for the starred alternatives. In that case, the resulting expected scores are:

$$\begin{aligned} \widehat{s}(b) &= 45 + 10 = 55, \\ \widehat{s}(a) &= 45, \\ \widehat{s}(c) &= 10 + 20 + 10 = 40, \\ \widehat{s}(d) &= 20. \end{aligned}$$

The reader can check that the described ballots (voters vote for the starred alternatives) are in equilibrium if the number of allowed votes is at least two per voter ($V \geq 2$). For example, consider a voter i with the first ranking. If she expects other voters to vote for the starred alternatives, her anticipations about expected scores are as follows:

$$\begin{aligned} \widehat{s}_{-i}(b) &= 44 + 10 = 54, \\ \widehat{s}_{-i}(a) &= 44, \\ \widehat{s}_{-i}(c) &= 10 + 20 + 10 = 40, \\ \widehat{s}_{-i}(d) &= 20. \end{aligned}$$

The weakest expected winner is candidate a and the strongest expected loser c. According to Step 3 in Proposition 11, the resulting order of priority for considering the candidates is the following: first, consider the two critical candidates (a and c), second, consider candidate b (whose distance to his main challenger c is $54 - 40 = 14$), third, consider candidate d (whose distance to his main challenger a is $44 - 20 = 24$). If $V \geq 2$, the strategic recommendation is to vote for a and b (preferred to c).

Now suppose that another candidate shows up but that the maximal number of votes is still set to two ($V = 2$). If the voters stick to the previous votes, the new candidate obtains at most 10 votes (from the last group of 10 voters, who still have one available approval), and this is an equilibrium for the constrained $V = 2$ voting rule. This remark holds true whatever the voters' preferences for the new candidate are. For instance this candidate could be the top choice of all the voters and still not be elected at this equilibrium.

It is not difficult to build such counter examples for any number V. The same coordination problem that is at play in the previous example has been described in theory in Myerson and Weber (1993) and Myerson (2002) for Plurality Rule in slightly different models ("Above the Fray"). The phenomenon seems robust, so that one can conclude that, at least in theory, the Restricted Approval voting rule suffers pathologies similar to those of the Plurality rule.

7 Empirics

7.1 The voting rule and the survey

The election of the seven members of the regional government in Zurich follows a voting rule which is almost exactly Approval Voting with a limit of seven on the number of votes. Notable differences are (i) the fact that there may be a second round in case not all seats are filled after the first round, and (ii) the fact that voters can approve of persons who are not listed as official candidates.

As to (i), a second round is organized if less than seven candidates reach a threshold of votes. In Zurich, the threshold is the total number of candidate votes cast (that is, individual approvals for candidates), divided by twice the number of seats. In practice, this is so low that it has never been the case in Zurich recent history that a second round was needed. Therefore (i) appears to be purely formal and can be neglected. As to (ii), the votes cast on non-candidates amount to about 8% in the election we will study now; but, because they are dispersed, these votes have no consequences, in theory nor in practice.

Therefore the Zurich election constitutes a case-study of $V = 7$-Restricted Approval Voting. In what follows, we take advantage of available individual data about preferences and votes to gauge the empirical relevance of the theory.

We study the election held on April 3, 2011. There were 9 registered candidates. Our analysis is based on data collected as part of the research project Making Electoral Democracy Work (Blais 2010). A survey was conducted on the occasion of the parliamentary and governmental elections. Respondents were asked, amongst other things, their votes and their evaluations of the candidates, through the following question:

> *Please rate each of the following candidates on a scale from 0 to 10, where 0 means that you strongly dislike the candidate and 10 means that you strongly like the candidate.*

A total of 502 respondents fully answered the vote and evaluation question. They form our data basis.[7]

[7] The survey was conducted on line by Harris International, relying on a panel from the

7.2 Empirical method

Following the theoretical developments of the previous sections, two things are needed in order to predict what a strategic voter should vote: (i) the anticipation, by the voter, of the candidate scores, and (ii) her ordinal preference about candidates.

As to (i), we will assume rational expectations and take as expected scores the final scores for this election.

As to (ii), we have the relevant information through the provided evaluations, except when the voter gives the same grade to two candidates who have to be compared. In that case, we give equal chance to each comparison.[8]

Table 1 provides key information about this election and the elements needed, according to our theoretical model, to compute each voter's rational response. The weakest expected winner is M. Graf and the strongest expected loser is H. Hollenstein.

Through an *ad hoc* program, we computed, for each of the 502 voters in our sample, the predictions derived from the model developed above. We can then aggregate these individual predictions, and compute the predicted scores of each official candidate (if voters were to react to official scores). Figure 1 compares, for each official candidate, his or her observed score (in our sample) with his or her predicted score (for our sample). One can see that the strategic model performs quite well in explaining the electoral scores observed in the sample.

The average number of approval per ballot (out of seven possible) is 4.39 in the sample and 4.23 in the predictions. More finely, Figure 2 depicts the distribution of the number of votes per ballot, in the observed sample, and according to the predictions. One can see that the main mistake done by the model is to underestimate the number of voters who cast a full ballot (7 candidates in that case). The model predicts that 12.3% of the ballots should approve 7 candidates but in reality, nearly twice more do so.

At the individual level, we have 502 respondents and 9 candidates so that we have 502*9=4518 vote predictions. On average the percentage of correct prediction is 69%, and is similar for positive (we predict that a voter approves a candidate) and negative (we predict that a voter does not approves a candidate) predictions. But the reliability of the strategic model differs from one candidate to the other, between 59% and 78%. (See Lachat et al (2014) for more details.)

These observations suggest that the behavior of voters can be described in the following terms: Some voters approve seven candidates simply because there are seven positions to fill. Another part of the population votes in a smoother way, which is rather well captured by the model we have presented. Given that voters from the second group should also sometimes vote for 7 candidates (about 12.3% of them according to our figures), it comes that the voters of the first group might represent about 10% of the electorate.

Swiss polling firm Link. The sample was representative as to age, gender, and education level. A more detailed analysis of this election can be found in Lachat et al. (2014).

[8] Because this may happen several times, and for the sake of simplicity, what we did is to replicate each participant 100 times, breaking all ties randomly.

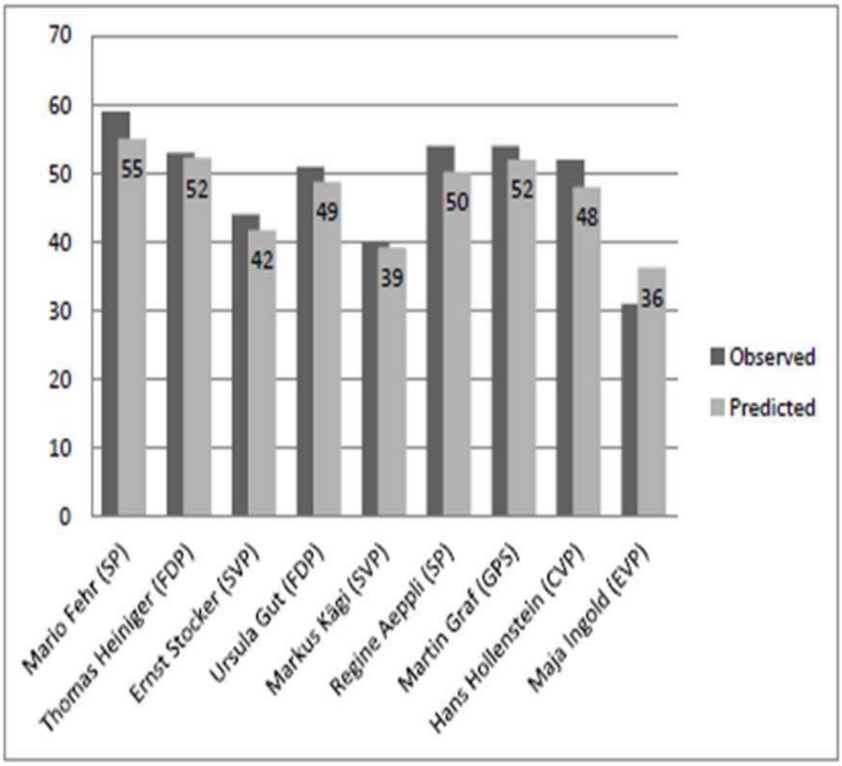

Figure 1: Electoral scores (% of voters in the sample)

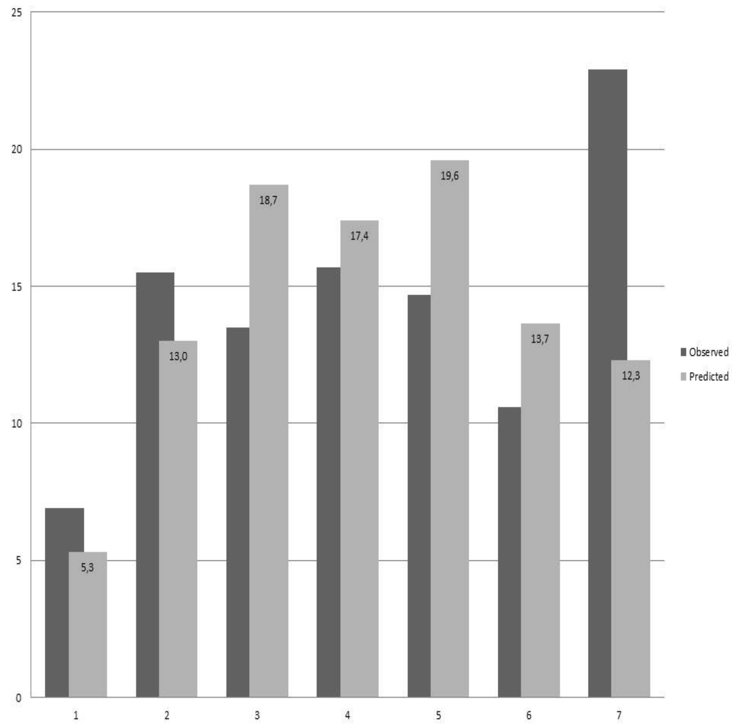

Figure 2: Distribution of the number of votes per ballot

(1)	(2)	(3)	(4)	(5)	(6)
Candidates	Number of votes	Rank in official election	Main contender	Distance to main contender	Priority order
M. Fehr	137035	1	Hollenstein	18548	7
T. Heiniger	134161	2	Hollenstein	15574	6
E. Stocker	129143	3	Hollenstein	11456	5
U. Gut	129349	4	Hollenstein	10862	4
M. Kagi	123159	5	Hollenstein	4672	3
R. Aeppli	121144	6	Hollenstein	2657	2
M. Graf	120815	7	Hollenstein	2328	1
H. Hollenstein	118147	8	Graf	2328	1
M. Ingold	68996	9	Graf	51819	8
Others	93485				

Table 1: Information needed to establish the strategic recommendation

8 Conclusion

We proposed a model of strategic voting in multi-winner elections with approval balloting. This model requires that the voters know their own preferences and evaluate the relative likelihoods of the possible electoral outcomes. It rests on a number of cognitive hypotheses: voters are only interested in the result of the election (no expressive motives), their have separable preferences, they are essentially rational, and they neglect three-way ties. All these hypotheses are questionable but they together have the virtue of producing definite predictions.

Equipped with these predictions, one can tackle positive and normative questions: Do people really behave like the model suggests? If yes, is it a good thing? The last section has shown that our purely theoretical model fits relatively well the actual behavior of the voters in one election we were able to study. It is therefore worth discussing what elements the theory can bring for a normative discussion of this voting rule.

We noticed that under "Unrestricted Approval" (no limit on the number of votes), the equilibrium properties of the model were very much in the spirit of an implementation of a generalization of the Condorcet principle to the case of a committee.

We found that, whatever $M \geq 1$, at equilibrium (if any), a candidate is elected if and only if it is supported by more than half of the electorate, and the Condorcet winner (if it exists) is part of the elected committee. Besides, when the majority tournament is transitive, there exists an equilibrium where the first M candidates according to the tournament are elected. This extends the finding by Laslier (2009) which showed that when $M = 1$, (Unrestricted) Approval Voting implements the Condorcet principle in the sense that strategic voters in equilibrium elect the Condorcet winner whenever it exists.

It should be noted that these properties are lost whenever there is a limit on the number of votes a voter is allowed to cast (Restricted Approval Voting). The example in section 6.2 highlights that, at least in theory, the Restricted Approval voting rule suffers from pathologies similar to that of the Plurality rule. In particular, it is prone to suffer from potential severe coordination problems. For instance, there can be situations where a candidate is the top choice of all the voters, and still is not elected at this equilibrium. From a normative point of view, "Unrestricted Approval Committee" seems more attractive than "Restricted Approval".

So far, when commenting in the paper upon the normative properties of various Approval Voting rules, the focus has been on their propensity to implement some Condorcet principle. Note that this criterion is not the only one one may have in mind when discussing the normative properties of rules designed to elect a committee. In particular, some concerns about the representativeness of the committee might be present. The following discussion will show that Approval Voting might perform quite poorly in that dimension.

Indeed, it should be highlighted that an important property of Approval Voting (and of the idea of a Condorcet winner) is lost when we go from $M = 1$ to $M > 1$.

Suppose that the same political party proposes, in a single-member district ($M = 1$), two candidates instead of one, and suppose that the preferences of the voters are such that the voters are chiefly interested in the parties, so that these two fellow candidates are ranked next to each other in every voter's preference. This manipulation[9] does not alter the fact that this party has a majority or not against an other party. In the very same manner, Unrestricted Approval Voting, by definition, lets the voter vote for several candidates if she wishes, and is thus immune to vote splitting or candidate duplication.

Now suppose that, in a district with $M > 1$ seats, all parties send M candidates, instead of only one. Then, under Unrestricted Approval Voting, the Condorcet-winning party on its own will gather all the seats, leading to a very poor representation of the different groups of voters in the electorate. In other

[9] This kinds of variation in the preference profile has a long history in Choice Theory; see the "Axiom 2.6" in Milnor (1951), the "Independence of Clones" in Tideman (1987), the "Composition-Consistency" in Laffond *et al.* (1996).

words, candidate duplication is ineffective under Approval Voting when $M = 1$ but is effective as soon as $M > 1$.

A Appendix

A.1 Proof of Lemma 1

Take as given the profile of strategies (ballots) of the voters, $s = (s_i)_{i=1,\dots,N}$. For any two candidates c and c',

$$\Pr\left[S(c) = S(c')\right] = \sum_{t=0}^{t=N} \Pr\left[S(c) = S(c') = t\right],$$

and, by independence:

$$\Pr\left[S(c) = S(c')\right] = \sum_{t=0}^{N} \left(\Pr\left[S(c) = t\right] \cdot \Pr\left[S(c') = t\right]\right).$$

Without loss of generality, assume that $\widehat{s}(c) \geq \widehat{s}(c')$.

Consider first the case where $t > \widehat{s}(c)$. The first order probability of the event $S(c) = t$ is

$$\begin{pmatrix} N - \widehat{s}(c) \\ t - \widehat{s}(c) \end{pmatrix} \varepsilon^{t - \widehat{s}(c)}. \tag{4}$$

Indeed, as one can easily check, the event $S(c) = t$ requires at least $t - \widehat{s}(c)$ mistakes, and can indeed result from that precise number of mistakes. One can and must pick $t - \widehat{s}(c)$ individuals who voted against c, among the $N - \widehat{s}(c)$ who voted against c, and change their votes to a YES vote in favor of candidate c. Thus the probability (4). A similar argument holds for the probability that c' get t votes, therefore, the first order probability of the event $S(c) = S(c') = t$ is:

$$\begin{pmatrix} N - \widehat{s}(c) \\ t - \widehat{s}(c) \end{pmatrix} \begin{pmatrix} N - \widehat{s}(c') \\ t - \widehat{s}(c') \end{pmatrix} \varepsilon^{2t - \widehat{s}(c) - \widehat{s}(c')}.$$

Similarly, when $t < \widehat{s}(c')$, the first order probability of the event $S(c) = t$ is

$$\begin{pmatrix} \widehat{s}(c) \\ \widehat{s}(c) - t \end{pmatrix} \varepsilon^{\widehat{s}(c) - t}$$

(pick $\widehat{s}(c) - t$ individuals among the $\widehat{s}(c)$ who voted for c, and change their votes to a NO vote for candidate c). Therefore the first order probability of the event $S(c) = S(c') = t$ is:

$$\begin{pmatrix} \widehat{s}(c) \\ \widehat{s}(c) - t \end{pmatrix} \begin{pmatrix} \widehat{s}(c') \\ \widehat{s}(c') - t \end{pmatrix} \varepsilon^{\widehat{s}(c) + \widehat{s}(c') - 2t}.$$

Last, when $\widehat{s}(c') \leq t \leq \widehat{s}(c)$, the first order probability of the event $S(c) = S(c') = t$ is :

$$\left(\begin{array}{c} \widehat{s}(c) \\ \widehat{s}(c) - t \end{array} \right) \left(\begin{array}{c} N - \widehat{s}(c') \\ t - \widehat{s}(c') \end{array} \right) \varepsilon^{\widehat{s}(c) - \widehat{s}(c')}.$$

When $t > \widehat{s}(c)$, $2t - \widehat{s}(c) - \widehat{s}(c') > \widehat{s}(c) - \widehat{s}(c')$ and when $t < \widehat{s}(c')$, $\widehat{s}(c) + \widehat{s}(c') - 2t > \widehat{s}(c) - \widehat{s}(c')$. Therefore, one can see that the event $S(c) = S(c')$ has first order probability:

$$\left(\sum_{t=\widehat{s}(c')}^{\widehat{s}(c)} \left(\begin{array}{c} \widehat{s}(c) \\ \widehat{s}(c) - t \end{array} \right) \left(\begin{array}{c} N - \widehat{s}(c') \\ t - \widehat{s}(c') \end{array} \right) \right) \cdot \varepsilon^{\widehat{s}(c) - \widehat{s}(c')},$$

so that the requirement of the event $S(c) = S(c')$ is $\widehat{s}(c) - \widehat{s}(c')$. **Q.E.D.**

A.2 Proof of Lemma 2

Given a profile of strategies $(s_i)_{i=1,\ldots,N}$, denote by S_M the random variable describing the M-th largest score obtained from the realized votes of all voters. Formally: for any vector of realized scores $(S(c))_{c\in\mathfrak{C}}$, let S_M be the unique number which satisfies the following two conditions:

1. $|\{c \in \mathfrak{C} : S(c) > S_M\}| \leq M - 1$,
2. $|\{c \in \mathfrak{C} : S(c) \geq S_M\}| \geq M$.

Candidates with scores strictly larger than S_M are elected, candidates with scores strictly smaller are not elected, and a candidate with score S_M is elected either for sure (if he is the only candidate with realized score S_M) or with some probability (in case of a tie with other candidates).

The event "Candidate c is caught in an exact tie for election" is the event "$S(c) = S_M$ and there exists at least one other $c' \neq c$ such that $S(c) = S(c') = S_M$".

Consider first the case where $k \leq M$. Let us show that the requirement of the event "$S(c_k) = S_M$ and there exists at least another $k' \neq k$ such that $S(c_k) = S(c_{k'}) = S_M$" is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$.

Note that $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ mistakes (from reference scores $\widehat{s}$) are sufficient to reach this outcome. Indeed, if out of the $\widehat{s}(c_k)$ voters who did vote for c_k, one picks $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ of them and change their votes (no other mistake being made), the resulting scores are $S(c) = \widehat{s}(c)$ for all $c \neq c_k$ and $S(c_k) = \widehat{s}(c_{M+1}) = S(c_{M+1})$. Note that this situation involves a two-way tie between candidate c_k and candidate c_{M+1}.

One can also check that any other vector of mistakes inducing that candidate c_k is caught in an exact tie for election implies at least as many mistakes.

Therefore the requirement of the event "$S(c_k) = S_M$ and there exists at least one other $k' \neq k$ such that $S(c_k) = S(c_{k'}) = S_M$" is exactly $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$.

Besides, one may check that the event "$S(c_k) = S_M$ and there exists at least one other $k' \notin \{k, M+1\}$ such that $S(c_k) = S(c_{k'}) = S_M$" (that is, not having candidate c_{M+1} part of the tie for the Mth position) involves strictly more mistakes.[10]

Consider now the case $k \geq M+1$. Let us show that the requirement of the event "$S(c_k) = S_M$ and there exists at least another $k' \neq k$ such that $S(c_k) = S(c_{k'}) = S_M$" is $\widehat{s}(c_M) - s(\widehat{s}_k)$.

Note that $\widehat{s}(c_M) - \widehat{s}(c_k)$ mistakes (from reference scores $\widehat{s}$) are sufficient to reach the outcome $S(c_k) = S(c_M) = S_M$. Indeed, if out of the $N - \widehat{s}(c_k)$ voters who did not vote for c_k, one picks $\widehat{s}(c_M) - \widehat{s}(c_k)$ and change their votes (no other mistakes being made), the resulting scores are $S(c) = \widehat{s}(c)$ for all $c \neq c_k$ and $S(c_k) = \widehat{s}(c_M) = S(c_M)$.

One can check that any other vector of mistakes inducing this outcome implies at least as many mistakes, therefore the requirement of the event "$S(c_k) = S_M$ and there exists at least one other $k' \neq k$ such that $S(c_k) = S(c_{k'}) = S_M$" is exactly $\widehat{s}(c_M) - \widehat{s}(c_k)$.

Besides, one may check that the event "$S(c_k) = S_M$ and there exists at least one other $k' \notin \{k, M\}$ such that $S(c_k) = S(c_{k'}) = S_M$" involves strictly more mistakes.[11] **Q.E.D.**

A.3 Proof of Proposition 3

Consider a profile of strategies (ballots) from voters other than voter i: $s_{-i} = (s_j)_{j \neq i}$. Let $\widehat{s}_{-i}$ denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i. Let the candidates be labelled in such a way that:

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > ... > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > ... > \widehat{s}_{-i}(c_K).$$

Assume that the expected vote difference between any two candidates is at least 3, that is, $\widehat{s}_i(c_k) - \widehat{s}_i(c_{k+1}) \geq 3$ for all $k = 1, , ..., K-1$.

To start the proof, consider a voter who contemplates any ballot s_i she could cast. Given the strategies s_{-i} of the other voters, the *ex post* utility that voter i derives from ballot s_i depends on the realization of the random variable ω

[10] Note nevertheless that there are events with requirement $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ where candidate c_k is caught in a tie for election with candidate c_{M+1} but also with another candidate. Indeed, consider an event where $\widehat{s}(c_k) - \widehat{s}(c_M)$ votes for c_k are not recorded, and where $\widehat{s}(c_M) - \widehat{s}(c_{M+1})$ NO votes for c_{M+1} are wrongly recorded as YES votes for c_{M+1}, no other mistake being made. The requirement of this event is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ and it involves a three-way tie for election between c_M, c_{M+1} and c_k. As mentionned in the description of the model (Section 2), we assume that the voter neglects this type of events involving three-way ties.

[11] Here again, note that there exists an event with requirement $\widehat{s}(c_M) - \widehat{s}(c_k)$ involving a three-way tie for election between c_M, c_{M+1} and c_k. We assumed that the voter neglects this type of events involving three-way ties.

describing the mistakes made when recording the ballots (remember $\omega_{j,c} = 1$ means that a mistake is made when recording voter j's vote about candidate c, see section 3). Denote this *ex post* utility by $U_i(s_i, s_{-i}, \omega)$. The expected utility derived from strategy s_i is $\sum_\omega U_i(s_i, s_{-i}, \omega) \Pr[\omega]$.

Consider two ballots, s_i and s_i',the voter prefers s_i to s_i' if and only if

$$\Delta = \sum_\omega U_i(s_i, s_{-i}, \omega) \Pr[\omega] - \sum_\omega U_i(s_i', s_{-i}, \omega) \Pr[\omega] \geq 0.$$

Obviously all the elementary events ω such that $U_i(s_i, s_{-i}, \omega) = U_i(s_i', s_{-i}, \omega)$ cancel in this inequality so that the sum can run over elementary events such that $U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega)$. This remark, with the fact that the probabilities $\Pr[\omega]$ are polynomials in ε (the requirement of event ω being $|\omega|$), provides the technique for finding best responses to an expected score vector $\widehat{s}_{-i}$ when ε is small. Let m be the requirement of the event $U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega)$. Then:

$$\begin{aligned} \Delta &= \sum_{\substack{\omega: U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega) \\ |\omega| = m}} [U_i(s_i, s_{-i}, \omega) - U_i(s_i', s_{-i}, \omega)] \Pr[\omega] \\ &+ \sum_{\substack{\omega: U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega) \\ |\omega| > m}} [U_i(s_i, s_{-i}, \omega) - U_i(s_i', s_{-i}, \omega)] \Pr[\omega]. \end{aligned}$$

The first part, where the sum runs over elementary events ω with requirement m, is a polynomial in ε of leading term $G\varepsilon^m$, where

$$G = \sum_{\substack{\omega: U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega) \\ |\omega| = m}} [U_i(s_i, s_{-i}, \omega) - U_i(s_i', s_{-i}, \omega)]$$

does not depend on ε.

The leading term of the second part has a strictly higher exponent, hence $G = \lim_{\varepsilon \to 0} \Delta \varepsilon^{-m}$.

It follows that, for ε small enough, the sign of Δ is the sign of G if $G \neq 0$. This implies that, in order to know whether s_i yields larger expected utility than s_i', one can restrict attention to those events which realize $U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega)$ with the smallest number of mistakes. Those events will involve ties (or near ties, with a one vote margin) for election of some candidates.

Given s_{-i}, what are the ballots s_i and s_i' and the events ω which realize $U_i(s_i, s_{-i}, \omega) \neq U_i(s_i', s_{-i}, \omega)$?

A necessary condition is that the ballots s_i and s_i' differ on a candidate which is caught in a tie (or a near tie) for election. Under our assumption that the voters in their computation of best responses neglect the possibility of three-way ties, we will focus on ties and near ties which involve exactly two candidates. Two candidates are said to be caught in an exact tie for election if the realized scores, given the votes of all the voters other than i, are such that both candidates receive the Mth highest score; they are said to be caught in a near tie for election if realized scores given the votes of all the voters other than i, are such that one of the candidate get the M-th highest score and the other candidate exactly one less vote. In both types of events, by voting for one of these candidates but not for the other, voter i can change the outcome of the election. Note that the difference between requirement of a tie and requirement of a near tie, for any given two candidate, is at most two.

Now, what are the events and ballots which realize $U_i(s_i, s_{-i}, \omega) \neq U_i(s'_i, s_{-i}, \omega)$ with the smallest number of mistakes?

Lemma 2 provides the answer. A straightforward adaptation of Lemma 2 states that, given the strategies s_{-i} of all voters but i, the requirement of the event "Candidate c_k is caught in an exact tie for election (not taking into account the vote of voter i)" is $\widehat{s}_{-i}(c_k)-\widehat{s}_{-i}(c_{M+1})$ if $k \leq M$ and $\widehat{s}_{-i}(c_M)-\widehat{s}_{-i}(c_k)$ if $k \geq M+1$. Therefore, the most likely exact tie for election occurs between candidate c_M (the weakest expected winner) and candidate c_{M+1} (the strongest expected loser), since the requirement of this event is $\widehat{s}_{-i}(c_M) - \widehat{s}_{-i}(c_{M+1})$. Given our assumption that the expected vote difference between any two candidates are at least 3, the most likely *near* tie (that is, with a one vote margin) for election also occurs between candidates c_M and c_{M+1}. Therefore, if voter i is pivotal, it will most likely be in deciding who between candidate c_M and candidate c_{M+1} will be elected. Therefore, if she prefers candidate c_M to candidate c_{M+1} ($u_i(c_M) > u_i(c_{M+1})$), she should vote for candidate c_M and not vote for candidate c_{M+1}. Similarly, if $u_i(c_M) < u_i(c_{M+1})$, she should vote for candidate c_{M+1} and not vote for candidate c_M. Her choice about candidates c_M and c_{M+1} is thus decided by this pairwise comparison between the two candidates.

What is the next most likely pivot-type event, involving at least one candidate other than candidate c_M and candidate c_{M+1}?

Again, Lemma 2 provides the answer. It will be either a tie (or near tie) for election between c_{M-1} and c_{M+1}, or a tie (or near tie) for election between c_M and c_{M+2}, depending on whether $\widehat{s}_{-i}(c_{M-1}) - \widehat{s}_{-i}(c_{M+1})$ is smaller or larger than $\widehat{s}_{-i}(c_M) - \widehat{s}_{-i}(c_{M+2})$. More generally, the results in Lemma 2 allow us to rank the different two-way ties for election involving candidates other than candidates c_M and c_{M+1}. Most specifically, if $1 \leq k \leq M$, define candidate c_k's "main contender" as c_{M+1} and if $M+1 \leq k \leq K$, define c_k's "main contender" as c_M. Then, rank the candidates according to (the inverse of) their distance, in terms of expected votes, to their main contender. As seen above, candidates c_M and c_{M+1} share the first rank in this ordering.

Consider now the candidate with the second position (either c_{M-1} or c_{M+2}), call this candidate $c(2)$. The next most likely pivot-type event involves a tie (or a near tie) between $c(2)$ and its main contender. Therefore, the voter should vote for $c(2)$ if and only if she prefers $c(2)$ to its main contender. Remember that the vote for or against $c(2)$'s main contender (c_M or c_{M+1}) has already been decided by the pairwise comparison between candidates c_M and c_{M+1}. Indeed the event "Candidate $c(2)$'s main contender is caught in a tie for election with candidate $c(2)$" is much less likely than a tie for election between c_M and c_{M+1}.

What is the next most likely pivot-type event, involving at least one candidate other than candidates $c(2)$, c_M and c_{M+1}? Denoting by $c(k)$, for $2 \leq k \leq K-1$ the candidate with the k's position in the ordering defined in the previous paragraph, one may check that the next most likely pivot-type event, involving at least one candidate other than candidates $c(2)$, c_M and c_{M+1} is a tie (or a near tie) between $c(3)$ and its main contender. Therefore, the voter should vote for $c(3)$ if and only if she prefers $c(3)$ to its main contender.

The same reasoning can be generalized by considering all the candidates in turn. Thus the strategic recommendation described in Proposition 3.

Q.E.D.

Remark 12 *In Section 6, we tackle the rule called "V-restricted Approval", whereby a voter can only approve up to V candidates. Note that the proof above also characterizes the best responses in that case. Indeed, in that case, the voter considers all the candidates in turn, according to the priority order defined in the proof. Note that the assumption that for any pair of candidates* (c, c'), $|\widehat{s}_{-i}(c) - \widehat{s}_{-i}(c')| \geq 3$ *in Proposition 11 guarantees that there is no ambiguity when defining this priority order. As long as she does not hit the vote-budget constraint (V votes), the voter votes for a candidate if and only if her utility for this candidate is larger than her utility for its main contender.*

A.4 Proof of Proposition 4

Consider a profile of strategies (ballots) from voters other than voter $i : s_{-i} = (s_j)_{j \neq i}$. Let $\widehat{s}_{-i}$ denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i. Let the candidates be labelled in such a way that:

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > ... > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > ... > \widehat{s}_{-i}(c_K).$$

- For $M = 1$ (one person to be elected), the best response described in Proposition 3 prescribes (i) to identify the critical candidates (c_1 and c_2), (ii) to approve c_1 if and only if $u_i(c_1) > u_i(c_2)$, (iii) for $k \geq 2$, to approve c_k if and only if $u_i(c_k) > u_i(c_1)$. This recommendation prescribes voting for all candidates strictly preferred to c_1 if $u_i(c_1) < u_i(c_2)$, and voting for all candidates weakly preferred to c_1 if $u_i(c_1) > u_i(c_2)$. This always produces a sincere ballot, whatever the voter's preferences over the candidates. This property for $M = 1$ was already noticed in Laslier (2009).

- For $M = K - 1$, this rule always produces a sincere ballot. Indeed, if $u_i(c_M) > u_i(c_{M+1}) = u_i(c_K)$: for any candidate c, she should vote for c if and only if she strictly prefers c to c_K. This always produces a sincere ballot. If $u_i(c_M) < u_i(c_{M+1})$: the voter should vote for a candidate c if and only if she weakly prefers c to c_K. This always produces a sincere ballot.

- Whenever $M \geq 2$ and $K \geq M+2$, there exist preferences for voter i such that strategic voting entails casting a non-sincere ballot. Suppose that voter i has preferences over the candidates such that:

$$u_i(c_M) > u_i(c_K) > u_i(c_1) > u_i(c_{M+1}),$$

  which is possible whenever $M \geq 2$ and $K \geq M+2$. The voter should approve the expected winners $(c_1, c_2, ..., c_M)$ if and only if she prefers them to the strongest expected loser c_{M+1}: given her preferences, this implies in particular voting for c_1. She should approve the expected losers $(c_{M+1}, c_{M+2}, ..., c_K)$ if and only if she prefers them to the weakest expected winner c: given her preferences, this implies in particular not voting for c_K. One concludes that such a voter should approve c_1 but not c_K, although she prefers c_K to c_1. This results in a non-sincere ballot.

Q.E.D.

A.5 Proof of Proposition 10

Assume voters have single peaked preferences.

In the single-peaked case, the majority tournament is transitive. By Proposition 9, there is an equilibrium where the M top candidates according to the majority tournament are elected. (Point 1 in Proposition 10).

By Remark 6, we also know that at any equilibrium, the Condorcet winner is elected.

Let us now show that the elected committee forms a segment in the ordered set of candidates.

Consider an equilibrium (by Proposition 9, we know that (at least) one equilibrium exists). Denote by x_M (resp. x_{M+1}) the position of the weakest winner in the ordered set of candidates (resp., the position of the strongest loser in the ordered set of candidates). Without loss of generality, assume that in the ordered set of candidates, $x_M < x_{M+1}$. Consider a candidate (assuming there is one), whose position in the set of ordered candidates is x, such that $x_M < x < x_{M+1}$. Let us show that necessarily, x is an expected winner (with a slight abuse of language, we will use in the sequel the same notation to denote a candidate, and its position in the ordered set of candidates). Indeed, assume by contradiction that x is an expected loser. By Proposition 5 Point 3, it must by the case that

$$N(x, x_M) < N(x, x_{M+1}).$$

Since preferences are single peaked and $x_M < x < x_{M+1}$, it must be the case that

$$N(x, x_M) > N(x_{M+1}, x_M),$$

yielding a contradiction. Therefore, any candidate located between x_{WW} and x_{SL} (if any) must be an expected winner.

Consider now a candidate, say x, (assuming there is one) such that $x < x_M$. Let us show that necessarily, x is an expected loser. Indeed, assume by contradiction that x is an expected winner. By Proposition 5 Point 2, it must by the case that

$$N(x, x_{M+1}) > N(x_M, x_{M+1}).$$

Since preferences are single peaked and $x < x_M$, it must be the case that

$$N(x, x_{M+1}) < N(x_M, x_{M+1}),$$

yielding a contradiction. Therefore, any candidate located on the left-hand side of x_M (if any) must be an expected loser.

Consider last a candidate, say x, (assuming there is one) such that $x > x_{M+1}$. Let us show that necessarily, x is an expected loser. Indeed, assume by contradiction that x is an expected winner. By Proposition 5 Point 2, it must by the case that

$$N(x, x_{M+1}) > N(x_M, x_{M+1}).$$

By Proposition 5 Point 3, we know that

$$N(x_M, x_{M+1}) > N/2,$$

therefore, it must by the case that the following two inequalities simultaneously hold:

$$\begin{aligned} N(x, x_{M+1}) &> N/2, \\ N(x_M, x_{M+1}) &> N/2, \end{aligned}$$

with $x_M < x_{M+1} < x$, yielding a contradiction with the fact that preferences are single-peaked. Therefore, any candidate located on the right-hand side of x_{M+1} (if any) must be an expected loser.

These remarks show that the set of expected winners forms a segment (in the set of ordered candidates). This concludes the proof of Point 2

To conclude the proof, since the Condorcet winner belongs to the set of the expected winners, and the set of winners is a segment,at most M distinct sets of winners can be supported at equilibrium.

Q.E.D.

Comment on Proposition 10: An example with single-peaked preferences where M distinct sets of winners can be elected at equilibrium. Proposition 10 states that if voters have single-peaked preferences, there are at most M distinct set of elected candidates which can be supported at equilibrium. We provide below a simple example showing that this maximum number can be reached.

Assume that there is a continuum of voters, with bliss points uniformly distributed on the interval $[-1, +1]$.

A voter with bliss point x^* evaluates a candidate at position x with the utility function $u(x, x^*) = -|x - x^*|$.

Assume M candidates are to be elected among $K = 2M$ candidates. Candidates are located as follows:

$$\begin{aligned} -1 &= l_M < l_{M-1} < ... < l_2 < l_1 < -0.5; \\ &+ \\ 0.4 &= x_1 < x_2 < ... < x_{M-1} < x_M < +0.5. \end{aligned}$$

The M top candidates according to the majority tournament are x_1, x_2, ..., x_M (ranked in that order from the Condorcet winner, x_1, to the M^{th} candidate in the tournament, x_M). The remaining candidates are l_1, l_2, ..., l_M, ranked in that order from the $(M+1)^{\text{th}}$ candidate in the tournament, l_1, to the Condorcet loser, l_M.

Necessary and sufficient condition for the set of winners to be the set

$$\{l_{M-k}, x_{M-k-1}, ..., , l_2 l_l, x_1, x_2, ...x_{k-1}, x_k\},$$

with x_k as the weakest winner and l_{M-k+1} as the strongest loser are (see Proposition 5):

Condition 1:	$N(x_k, l_{M-k+1}) > 1/2$,
Condition 2a:	$N(l_j, l_{M-k+1}) > N(x_k, l_{M-k+1})$ for $j < M - k + 1$
Condition 2b:	$N(x_j, l_{M-k+1}) > N(x_k, l_{M-k+1})$ for $j < k$ (if $k \geq 2$)
Condition 3a:	$N(l_j, x_k) < N(l_{M-k+1}, x_k)$ for $j > M - k + 1$
Condition 3b:	$N(x_j, x_k) < N(l_{M-k+1}, x_k)$ for $j > k$

It is straightforward to check that Conditions 1, 2a, 2b, 3a are clearly satisfied. Condition 3b is satisfied if and only if:

$$N(x_{k+1}, x_k) < N(l_{M-k+1}, x_k).$$

Note that:

$$N(l_{M-k+1}, x_k) > N(l_M, x_1) = 35\%,$$

and

$$N(x_{k+1}, x_k) < N(x_2, x_1) < 30\%.$$

Therefore $N(x_{k+1}, x_k) < N(l_{M-k+1}, x_k)$, and Condition 3b is satisfied.

This simple example shows that for any $k = 1, 2, ..., M$, the set of candidates $\{l_{M-k}, x_{M-k-1}, ..., , l_2, l_1, x_1, x_2, ...x_{k-1}, x_k\}$ can be elected at equilibrium. This proves that there can be M distinct sets of winners at equilibrium, including one where the set of elected candidates is $\{l_{M-1}, x_{M-2}, ..., , l_1, x_1\}$. In this equilibrium, note that among the M elected candidates, only the Condorcet winner belongs to the set of the M top candidates according to the majority tournament.

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